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An internal wave in a viscous heat-conducting isothermal atmosphere

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The effects of viscosity and heat conduction on the propagation of an internal wave generated by a simple harmonic localized disturbance are considered for the case of an isothermal compressible atmosphere. A similarity solution of the linearized equations shows that the velocities decay and the wave width increases away from the disturbance. Superpositions of this solution show how a few waves of small wavelength attenuate rapidly whereas waves of larger wavelength can increase in amplitude as they propagate upwards before eventually attenuating.

1. Introduction

The propagation of internal waves and acoustic waves in a stably stratified compressible non-dissipative atmosphere has been studied by Midgley & Liemohn (1966), Moore & Spiegel (1964), Tolstoy (1963) and many others (see the review by Hines 1972). Most of these authors are interested in the way in which a complete spectrum of wave frequencies transfers energy between various heights in the atmosphere. Estimates of the effects of viscosity and heat conduction were made by Hines (1960) and Pitteway & Hines (1963) and more detailed analyses were presented by Yanowitch (1967), who excluded heat conduction, Lindzen (1970, 1971) and Lindzen & Blake (1971). Lindzen's papers also include the effects of hydromagnetic drag and radiation in an atmosphere with arbitrary distributions of background temperature. In these analyses the linearized equations are reduced to ordinary differential equations by considering a wave form which is sinusoidal in the horizontal co-ordinate. When the frequency of oscillation is less than the natural frequency,† Lindzen's solutions for the isothermal atmosphere show that the wave amplitude increases with altitude before eventually tending to a constant value. For frequencies which are higher than the natural frequency the amplitudes decrease exponentially.

The present note studies viscous and heat-conduction effects in a two-dimensional internal wave produced by a localized disturbance of fixed frequency in an isothermal atmosphere. The frequency is less than the natural frequency and the waves are not sinusoidal in the horizontal direction. The solution is based on

† That is, the buoyancy, or Brunt–Väisälä frequency.

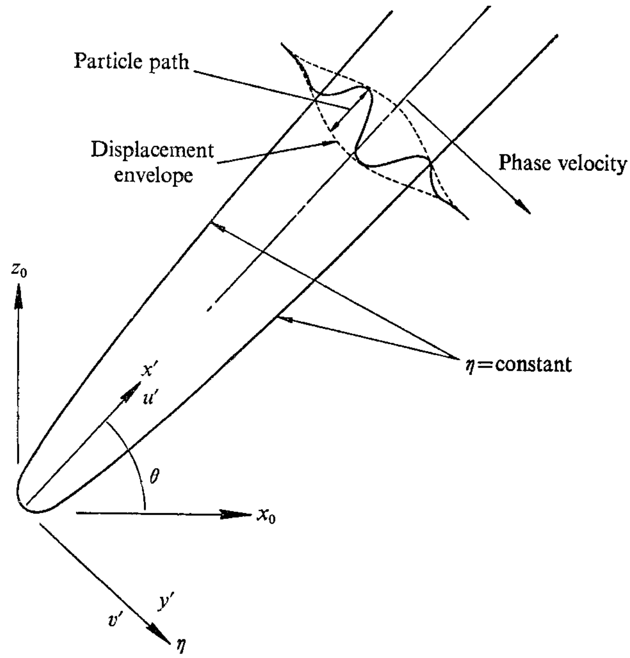

FIGURE 1. Co-ordinate axes.

that of Thomas & Stevenson (1972) for the waves produced by an oscillating body in an incompressible stratified fluid. It will be shown how, as the energy propagates upwards, the wave increases in width and decreases in amplitude.

2. Theory

A horizontal two-dimensional disturbance of frequency ω in an unbounded isothermal atmosphere will be considered. The disturbance is near the origin of a Cartesian co-ordinate system Ox_0z_0, which is fixed relative to the undisturbed background fluid, with z_0 measured vertically upwards. It is assumed that the atmosphere is a perfect gas, that the acceleration g due to gravity is constant and that the effects of rotation are unimportant. Radiation and humidity are not included in the analysis.

β, the inverse of the stratification scale height, is given by

$$\beta = -\frac{1}{\rho_0}\frac{d\rho_0}{dz_0} - \frac{g}{c_0^2} = \frac{(\gamma-1)}{c_0^2}g, \tag{1}$$

where the subscript zero refers to the background conditions, ρ is the density, c is the speed of sound and γ is the ratio of the specific heats. The equation of state is $p = \rho RT$, where p is the pressure and T is the temperature.

A second co-ordinate system $Ox'y'$ has the x' axis along the wave and the y' axis in the direction of the inviscid phase velocity. Thus $x' = x_0\cos\theta + z_0\sin\theta$ and $y' = x_0\sin\theta - z_0\cos\theta$, where θ is the angle between the Ox' and the horizontal. The background hydrostatic equations are

$$\partial p_0/\partial x' = -\rho_0 g\sin\theta, \quad \partial p_0/\partial y' = \rho_0 g\cos\theta. \tag{2}$$

The velocity components are u' and v', the density, pressure and temperature are ρ_s, p_s and T_s and t' is the time. The perturbation variables are $\rho' = \rho_s - \rho_0$, $p' = p_s - p_0$, $T' = T_s - T_0$, $\mu' = \mu_s - \mu_0$ and $k' = k_s - k_0$, where μ_s is the viscosity and k_s is the thermal conductivity.

The equation of continuity is

$$\frac{D\rho_s}{Dt'} + \rho_s\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0, \tag{3}$$

where

$$\frac{D}{Dt'} = \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}.$$

The perturbation momentum equations after subtracting the hydrostatic relations are

$$\rho_s\frac{Du'}{Dt'} + \frac{\partial p'}{\partial x'} + \rho' g\sin\theta = \frac{\partial}{\partial x'}\left\{(2\mu_s + \lambda_s)\frac{\partial u'}{\partial x'} + \lambda_s\frac{\partial v'}{\partial y'}\right\} + \frac{\partial}{\partial y'}\left\{\mu_s\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\right\} \tag{4}$$

and

$$\rho_s\frac{Dv'}{Dt'} + \frac{\partial p'}{\partial y'} - \rho' g\cos\theta = \frac{\partial}{\partial x'}\left\{\mu_s\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\right\} + \frac{\partial}{\partial y'}\left\{\lambda_s\frac{\partial u'}{\partial x'} + (2\mu_s + \lambda_s)\frac{\partial v'}{dy'}\right\}, \tag{5}$$

λ_s being the bulk viscosity. The energy equation is

$$\rho_s c_v\frac{DT'}{Dt'} + p_s\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = \frac{\partial}{\partial x'}\left(k_s\frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k_s\frac{\partial T'}{\partial y'}\right) + \Phi, \tag{6}$$

where Φ is the viscous dissipation.

In an isothermal atmosphere the hydrostatic equations (2) and the equation of state give

$$\frac{p_0}{p^*} = \frac{\rho_0}{\rho^*} = \exp\left\{-\frac{g}{RT_0}(x'\sin\theta - y'\cos\theta)\right\}, \tag{7}$$

where the starred variables are constant reference conditions at $x' = y' = 0$.

New (undashed) dimensionless variables are introduced as follows:

$$t' = t(N\sin\theta)^{-1}, \quad x' = x(\beta\sin\theta)^{-1}, \quad y' = y\alpha(\beta\sin\theta)^{-1}, \quad u' = uagN^{-1},$$
$$v' = va\alpha gN^{-1}, \quad \rho' = \rho a\rho^*, \quad p' = pa\alpha\rho^* g(\beta\tan\theta)^{-1}, \quad T' = TaT_0,$$
$$\mu' = \mu a\mu^*, \quad k' = kak^*$$

and the particle displacement from the equilibrium position $\xi' = \xi a(\beta\sin\theta)^{-1}$. Also $\alpha^3 = N^3\nu^*\tan\theta\sin\theta/2g^2$ and $\nu^* = \mu^*/\rho^*$. a is an amplitude coefficient which is constant and $N = (\beta g)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency.

The analysis now follows the incompressible solution (Thomas & Stevenson 1972) and will merely be outlined. It is assumed that θ must not be near 0 or $\frac{1}{2}\pi$. The limitations introduced by these approximations will be discussed later.

Equations (3)–(6) reduce to

$$\frac{\partial\rho}{\partial t} = r_0\left[\frac{\gamma}{\gamma-1}(u - v\alpha\cot\theta) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right], \tag{8}$$

$$r_0\frac{\partial u}{\partial t} + \alpha\cot\theta\frac{\partial p}{\partial x} + \rho = 2\alpha\cot\theta\frac{\partial}{\partial y}\left(\frac{\mu_0}{\mu^*}\frac{\partial u}{\partial y}\right) + O(\alpha^2), \tag{9}$$

$$r_0\alpha\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}\cos\theta + \rho\cot\theta + O(\alpha^2) \tag{10}$$

and
$$\frac{\partial T}{\partial t}+(\gamma-1)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{2\gamma}{\sigma^* r_0}\alpha\cot\theta\frac{\partial}{\partial y}\left(\frac{k_0}{k^*}\frac{\partial T}{\partial y}\right)+O(\alpha^2), \tag{11}$$

where $r_0=\rho_0/\rho^*=p_0/p^*$ and the Prandtl number $\sigma^*=\mu^*c_p/k^*$. The equation of state becomes

$$r_0T+\rho=\alpha\chi p, \tag{12}$$

where
$$\chi=\frac{\rho^* g\cot\theta}{p^*\beta}=\frac{\gamma}{\gamma-1}\cot\theta.$$

We look for a solution in which the perturbation variables have a time dependency e^{-it}. The boundary conditions are that u', v', p', ρ' and T' and their derivatives should approach zero as $y'\to\pm\infty$. We assume that the variables may be expanded as $u=u_1+\alpha u_2+\ldots$, $v=v_1,+\alpha v_2+\ldots$, $p=p_1+\alpha p_2+\ldots$, etc. r_0 may be written as $r_0=r_1(x)+\alpha r_2(x,y)+\ldots$, where from (7)

$$r_1=\exp\left(-\frac{\gamma x}{\gamma-1}\right),\quad r_2=\frac{\gamma y}{\gamma-1}\cot\theta\exp\left(-\frac{\gamma x}{\gamma-1}\right).$$

Similarly we write

$$\mu_0/\mu^*=m_1(x)+\alpha m_2(x,y)+\ldots\quad\text{and}\quad k_0/k^*=n_1(x)+\alpha n_2(x,y)+\ldots.$$

These expansions are substituted into (8)–(12) and terms of like order are equated (see Thomas & Stevenson 1972). The resulting equation for p_1 is

$$i\frac{\partial}{\partial x}(p_1r_1^{-\frac{1}{2}})=\frac{1}{r_1}\left(m_1+\frac{n_1}{\sigma^*}\right)\frac{\partial^3}{\partial y^3}(p_1r_1^{-\frac{1}{2}}). \tag{13}$$

If the viscosity and thermal conductivity are assumed to be functions of temperature only so that $m_1=n_1=1$, then (13) can be written as

$$\partial^3(p_1r_1^{-\frac{1}{2}})/\partial y^3-i\partial(p_1r_1^{-\frac{1}{2}})/\partial X=0, \tag{14}$$

where
$$X=\left(1+\frac{1}{\sigma^*}\right)\left(\frac{\gamma-1}{\gamma}\right)\left(\exp\left(\frac{\gamma x}{\gamma-1}\right)-1\right).$$

This now has the incompressible form and a similarity solution satisfying the boundary conditions and a constant momentum flux condition is

$$p_1r_1^{-\frac{1}{2}}=\mathscr{R}\{X^{-\frac{1}{3}}f(\eta)\,e^{-it}\}, \tag{15}$$

where
$$f=\int_0^\infty\exp(-\kappa^3)\exp(i\kappa\eta)\,d\kappa,\quad \eta=y/X^{\frac{1}{3}}. \tag{16}$$

The dimensionless variables take the following forms:

$$u_1=\mathscr{R}\{-ir_1^{-\frac{1}{2}}X^{-\frac{2}{3}}(df/d\eta)\,e^{-it}\}, \tag{17}$$

$$\rho_1=-r_1T_1=\mathscr{R}\{r_1^{-\frac{1}{2}}X^{-\frac{2}{3}}(df/d\eta)\,e^{-it}\} \tag{18}$$

and the dimensionless particle displacement is

$$\xi=\mathscr{R}\{r_1^{-\frac{1}{2}}X^{-\frac{2}{3}}(df/d\eta)\,e^{-it}\}. \tag{19}$$

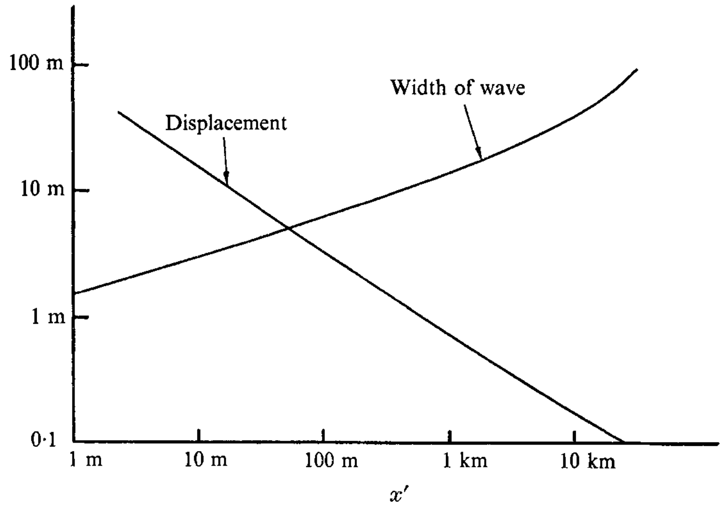

FIGURE 2. The variation of the maximum displacement within the wave and the width of the wave with distance from the origin. The width of the wave was evaluated between $\eta = \pm 6$. $c_0 = 295\,\mathrm{m\,s^{-1}}$. $\beta = 4{\cdot}5\times10^{-5}\,\mathrm{m^{-1}}$, $\alpha = 0{\cdot}8\times10^{-4}$, $a = \frac{1}{10}\alpha$, $\sigma^* = 0{\cdot}73$, $\theta = 45°$, $N = 2{\cdot}1\times10^{-2}\,\mathrm{rad\,s^{-1}}$. The values are taken from the International Standard Atmosphere and the origin $x' = 0$ is at 11 km.

The phase velocity is in the direction of increasing η: the direction pointing towards the horizontal level of the disturbance. When (16) and (17) are written in terms of x it is seen that the velocity decays as

$$\left(\exp\left(\frac{\gamma x}{\gamma-1}\right)-1\right)^{-\frac{2}{3}}\exp\left(\frac{\gamma x}{2(\gamma-1)}\right)$$

and the wave width increases as

$$\left(\exp\left(\frac{\gamma x}{\gamma-1}\right)-1\right)^{\frac{1}{3}}.$$

The solution shows that the effects of thermal conduction and viscosity are of equal importance. The relative pressure perturbations are several orders of magnitude less than the relative density or temperature perturbations.

The solution is used to calculate the maximum displacement and the wave-width variation within an internal wave in the stratosphere and the result is shown in figure 2. For the stratosphere α is of order 10^{-4} and, therefore, the assumption that $\alpha \leqslant 1$ is justified. Thomas & Stevenson (1972) discuss the accuracy expected from the theory and using that analysis it is found that the major restriction which is important for atmospheric flows is that the nonlinear terms should be small compared with the linear terms. For the calculations of figure 2, the analysis indicates that the nonlinear terms are important when x' is less than 100 m. If a smaller value of α were used then the theory could be applied closer to the virtual origin of the disturbance, but the allowable amplitudes throughout the wave would be reduced. Note that a value of $x = 1$ corresponds to a real distance x' of the order of 10 km.

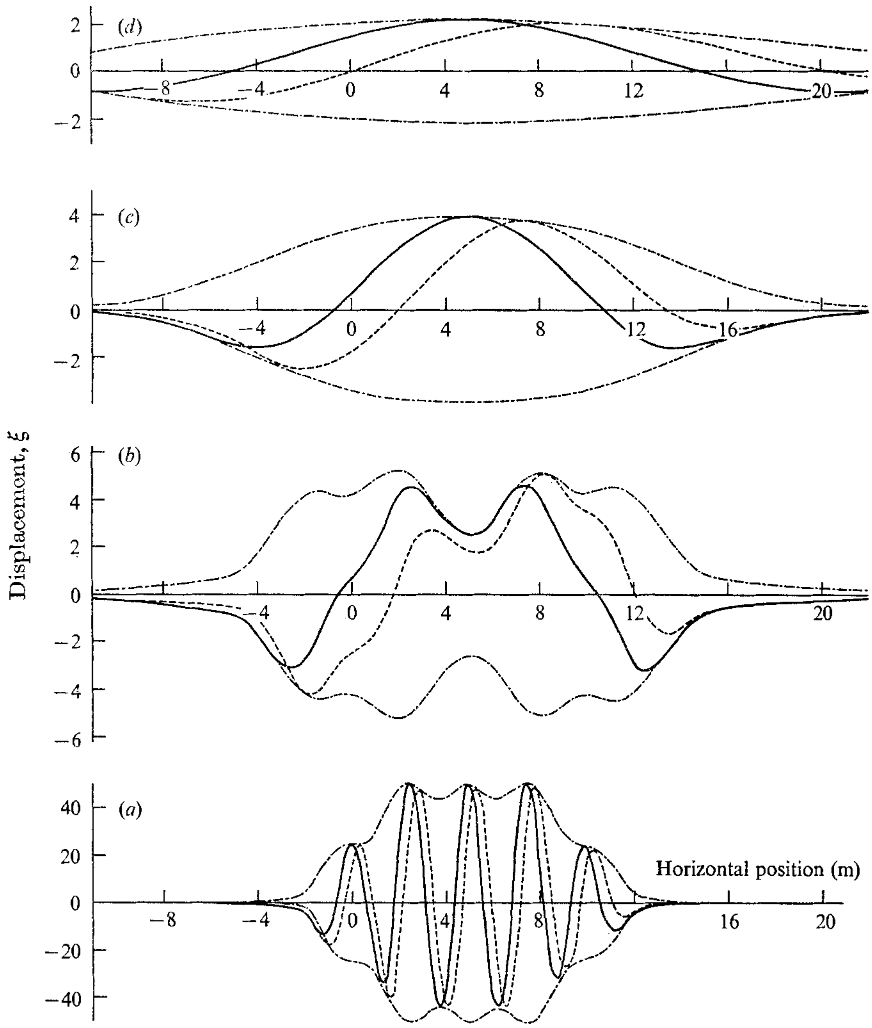

FIGURE 3. Displacement profiles on horizontal surfaces at various heights. The background stratification is the same as that in figure 2. The profiles were obtained by superimposing five waves which have origins on the same horizontal level with $x_0 = 0, 2{\cdot}5, 5{\cdot}0, 7{\cdot}5, 10{\cdot}0$ m. The waves are in phase, have $\theta = \frac{1}{4}\pi$ and have amplitudes in the ratio 0·5, 1·0, 1·0, 1·0, 0·5, respectively. The zero on each of the horizontal scales in the figure corresponds to $\eta = 0$ on the first wave and (a) $x' = 10$ m, (b) $x' = 100$ m, (c) $x' = 1$ km, (d) $x' = 10$ km. The zeros therefore lie on the line $x_0 = z_0$ (see figure 1). Displacement profiles: ——, $t = \frac{1}{2}\pi$; – – –, $t = \frac{3}{4}\pi$. — — —, displacement envelopes. The vertical height z_0 is given by $z_0 = 2\frac{1}{2}x'$.

More realistic wave forms are obtained by superpositions of this solution. The way in which a packet of waves from five disturbances propagates and changes in amplitude is shown in figure 3. As the waves propagate upwards they interact and eventually produce a profile of similar shape to that produced by an individual disturbance. The decrease in amplitude with height for this wave system is compared with that for an individual wave in figure 4 and also with that

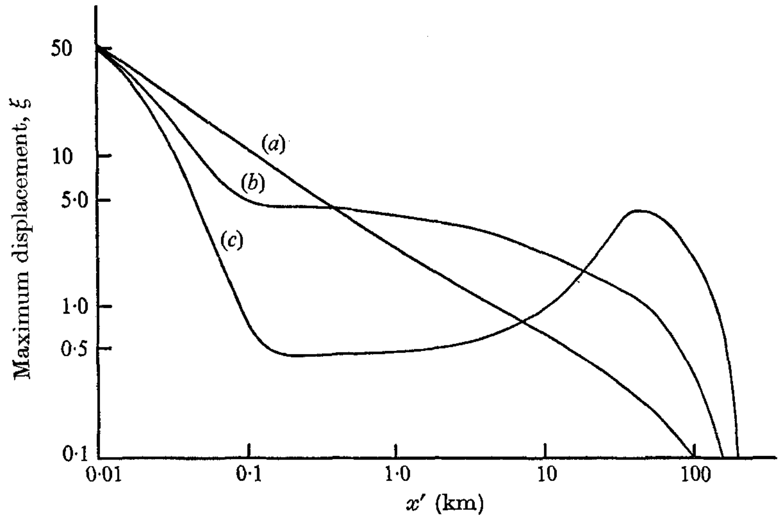

FIGURE 4. The way in which the maximum displacements in a wave system vary with altitude. The background conditions are the same as in figure 2. $\theta = \frac{1}{4}\pi$. (*a*) One wave. (*b*) The wave system of figure 3. (*c*) Maximum displacements in the central region of a wave system produced by 20 disturbances with the same frequency and amplitude and at 2·5 m spacing.

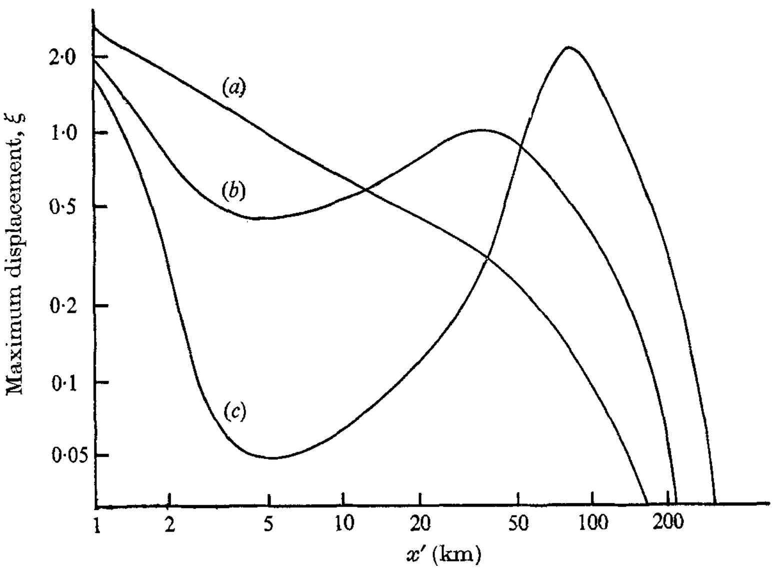

FIGURE 5. The same as figure 4 except that the wave spacing is now 9 m. (*a*) 1 wave. (*b*) 5 waves. (*c*) 20 waves. The displacement profile at $x' = 1$ km with 5 disturbances is the same shape as that in figure 3 (*a*) except that the wavelength is now 9 m instead of 2·5 m.

for the central region of a wave system produced by 20 disturbances each separated by 2·5 m. When there are many disturbances the central region of the wave system attenuates rapidly, leaving higher amplitudes of oscillation towards the outer edges of the system. As the energy propagates upwards the outer peaks

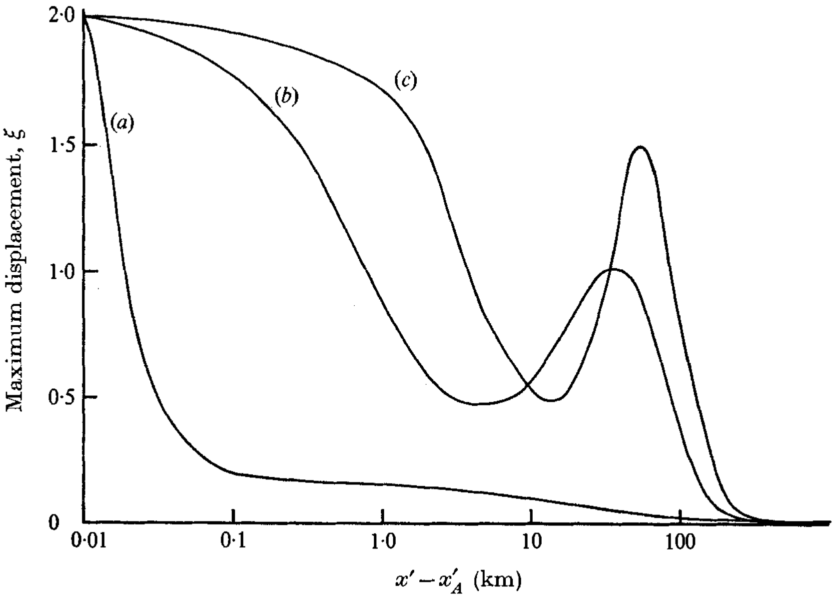

FIGURE 6. Variation of the maximum displacement with altitude. x'_A is chosen so that the waves are of the same shape as those in figure 3(*a*) at the position $x'-x'_A = 0{\cdot}1$ km. The wavelengths of the waves at this position are (*a*) 2·5 m, (*b*) 9 m, (*c*) 20 m. The background conditions are the same as those in figure 2.

spread and overlap in the central region and it is this effect which causes an amplitude increase over certain heights. Figure 5 shows the maximum displacements that occur if the disturbances are 9 m apart and in this case both the five- and the twenty-disturbance wave systems amplify over certain regions. In figure 6 it is shown how waves of small wavelength attenuate more rapidly than waves of longer wavelength.

Small amplitude inviscid solutions for the isothermal atmosphere suggested that there is an exponential increase of amplitude with height due to the decreasing air density, e.g. Lamb (1932). However, the viscous theory of Lindzen for a wave system stretching to infinity in the horizontal directions showed that the amplitudes after initially increasing tend to a constant value. It has now been shown how a wave packet consisting of a few waves can widen and eventually attenuate at sufficiently large altitudes.

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